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Convergence Theorems of Iterative Schemes for Non-expansive Mapping in Banach Space

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Abstract

In this paper, we propose and analyze a new class of iterative schemes designed to approximate the fixed point of T. The convergent behavior of the scheme is investigated under various geometric assumptions on the space X. Let X be a real Banach space and let $C \subseteq X$ be a nonempty closed convex subset. Consider a nonexpansive mapping $T: C \rightarrow C$ that is $\|T_{(x)} - T_{(y)}\| \leq \|x - y\|$ for all $x, y \in C$ under appropriate conditions on the control parameter. We show that the generated sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n)$$

In this work, we aim to establish a new fixed-point theorem for non-expansive and quasi-nonexpansive mappings in Banach spaces using the D:D. iterative process.

Introduction

Let E be a closed, convex and bounded subset of a Banach space X. and let $T: E \to E$ be a mapping. The mapping T is said to be expansive if it satisfies the condition

$$\|T_{(x)} - T_{(y)}\| \leq \|x - y\| \quad \forall\, x, y \in E$$

The fixed point set of T., denoted by $F(T) = \{x \in E: T(x) = x\}$, consists of all points in E that remain invariant under T. In the context of Hilbert space, Baillon, 1967, established that if E is a closed convex subset of a Hilbert space H and T has at least one fixed point, then the sequence $\left\{T^n_{(x)}\right\}$ converges weakly to a Fixed point of T as $n \to \infty$. This result, now known as the first ergodic theorem, has served as a Foundation for further development in the area. Pazy, 1977. later demonstrated that in a real Hilbert space H. If the sequence $\frac{1}{n}\sum_{j=0}^{n-1} Tj'_{(x)}$ Converges weakly to some y $\in E$ then y it must also be a fixed point of T. The Concept of Ouasi-Nonexpensive mapping on extension of nonexpansive mapping has it is origins in the work of Tricomi, 1941 for real-valued functions and formally developed in the setting of Banach space by researchers such as Doston 1970, Diaz and Metcalf,1969 , and later extended by Kirk,1997 and Olusegun,2006.

A mapping T is said to be Quasi-Quasi-Quasi-Quasi-Nonexpansive if for any fixed point $E \in F(T)$ $\|T_{(x)} - t\| \leqslant \|x - t\| \ \forall\, x \in F$

In recent developments, iterative schemes have been employed to approximate the fixed point of such a mapping. One such method is the D.D. iterative scheme, a modification of the iterative process introduced by Pathak, which also satisfies Opial's conditions.

Preliminaries

Let X be a Banach space and let E $\subseteq$ X be a nonempty, closed and convex subset. Consider three mappings T, S, R:E $\rightarrow E$, We introduce an extended form of the D.D. (Double Difference) iterative scheme that incorporates all three mappings. Given an initial point, $x \in E$ the sequence $\{x_n\} \subset E$ is generated according to the following iterative process:

$$x_{n+1} = \alpha_n T_{(x_n)} + \beta_n T_{(y_n)} + \gamma_n R_{x_n} \qquad (i)\ y_n = \alpha'_n S_{(x_n)} + \beta'_n R_{(x_n)} + \gamma'_n T_{x_n}$$

Where the real sequences $\{\alpha_n\}\{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ are chosen. Such that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in N$ similarly, the sequence $\{\alpha'_n\}, \langle\beta'_n\}\{y'_n\} \subset (0, 1)$ appearing in intermediate steps of the process are assumed to satisfy $\alpha'_n + \beta'_n + \gamma'_n = 1$. We also assume that the Banach space X satisfies Opial's condition, which plays a crucial role in establishing strong convergence of the sequence. Specifically, a Banach X satisfies Opial's Conditions if for every sequence $\{x_n\} \subset X$that convergence weakly to a point $x \in X$, the inequality

$\lim_{x\to\infty} \|x_n - x\| \leq \lim_{x\to\infty} inf \|x_n - y\| \, Holds$

For all $y \in x$ with $y \neq x$

The convergence behavior is both strong and weak if the iterative sequence for approximating a fixed point Quasi-Nonexpansive mapping has been extensively studied. Petyshyn 1978. investigated the behavior of Ruch sequence in the context of P fixed point theory building.

Upon earlier work by Doston, on the convergence of Mann iteration, further development was made by Ghosh and Debnath, 1997, who examined the convergence properties of the Isikawa type. iteration for Quasi-Nonexponential mapping. In the setting if Hilbert space, a weak convergence theorem for 1 - Asymptotically Quasi-Nonexpansive mapping was presented in Rhoades,2006.

Lemma

Let X be a Banach space and E ⊆ X be nonempty, closed, and Convex. Assume $T, S, R: E \to E$ are a nonexpansive mapping with a common fixed point $P \in E$. Then the sequence $\{x_n\}$ generated by the D. D. iteration is bounded.

Proof :-

Let $x_0 \in E$ and E is convex, all subsequent iterates $y_n, z_n, x_{n+1} \in E$

Fix $p \in Fix\ (T) \cap Fix\ (S) \cap Fix\ (R)$ then,

$\|y_n - p\| = \|(1 - \alpha_n)x_n + \alpha_n Rx_n - p\| \leqslant (1 - \alpha_n)(x_n - p) + \alpha_n(Rx_n - p)\| \leqslant (1 - \alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| = \|x_n - p\|$

$\|y_n - p\| \leq \|x_n - p\|$

similarly $\|Z_n - p\| \leqslant \|x_n - p\|$ and

$$\|x_{n+1} - P\| \leq \|x_n - P\|$$

Hence, $\|x_{n+1} - p\| \leq \|x_n - p\|$ which implies $\{\|x_n = p\|\}$ is non-increasing and bounded below, so it converges. Therefore, it $\{x_n\}$ is bounded.

Lemma

Under the assumptions of Lemma 1, the sequence $\{x_n\}$ generated by the D.D. Scheme satisfies

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$$

Lemma

Let $T: E \to E$ be a nonexpansive mapping

of $x_n \to x$ (weakly) and $\|T_{(x_n)} - x_n\| \to 0$ then $x \in$ Fix (T)

Lemma

Assume X satisfies. Opial's Condition If $x_n \to x$ and $x \in$

$Fix\ (T) \cap Fix\ (S) \cap Fix\ (R)$ then x is the unique weak limit of $\{x_0\}$.

Results and Discussion

Theorem

Let E be a nonempty, closed, and bounded subset of a uniform. A convex Banach space X that satisfies Opial's conditions.

Let $T,\ S,\ R : E \rightarrow E$ be nonexpansive self-mappings. Then, for an initial point $x_0 \in E$, the sequence $\{x_0\}$ generated by the D.D. iterative scheme converges weakly to a common fixed point of $F(T)\cap F(S)\cap F(R)$

Proof

Let X be a uniformly convex Banach space E $\subseteq$ X be nonempty, closed, bounded, and convex and T, S, R: $E \rightarrow E$ be nonexpansive mapping i.e. for all $x, y \in E$

$\|T_{(x)} - T_{(y)}\| \leqslant \|x - y\|$ and similarly for S and R

$$\|S_{(x)} - S_{(y)}\| \leq \|x - y\| \ and \ \|R_{(x)} - R_{(y)}\| \leq \|x - y\|.$$

then $F(T), F(S), F(R)$ denoted the. set of fixed points of T, S, R respectively.

Assume that the common fixed point set $F := Fix\ (T)\cap Fix\ (S)\cap Fix\ (R)$ is nonempty, then the iterative Sequence $\{x_n\}$ is defined by the D.D. scheme as

$$x_{n+1} = \alpha_n T_{(x_n)} + \beta_n T_{(x_n)} + \gamma_n R_{(x_n)}, \qquad n \geq 0$$

Where $\alpha_n, \beta_n, \gamma_n \in [0, 1]$, $\alpha_n + \beta_n + \gamma_n = 1$ and each sequence $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ satisfies

$$\lim_{n\to\infty} \alpha_n = \alpha > 0 \quad \lim_{n\to\infty} \beta_n = \beta > 0, \ \lim_{n\to\infty} \gamma_n = \gamma > 0$$

Now let $P \in F$ since $T(P) = S(P) = R(P)$ for each n

$$\|x_n - p\| = \|\alpha_n T_{(x_n)} - \beta_n S_{(x_n)} = \gamma_n R_{(x_n)} - P\|$$

$$Using\ convexsity\ and\ triagle\ inequelity$$

$$\|x_{n+1} - p\| \leq \alpha_n \|T_{(x_n)} - p\| + \beta_n \|S_{(x_n)})p\| + \gamma_n \|R_{(x_n)} - p\|$$

Since all three mappings are nonexpansive and P is a fixed point

$$\|T_{(x_n)} - P\| \leq \|x_n - P\|$$

$$\|S_{(x_n)} - p\| \leq \|x_n - p\|$$

$$\|R_{(x_n)} - p\|\| \leq \|x_n - p\|$$

There for $\|x_{n+1} - p\| \leq \alpha_n \|x_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\|$

$$= \|x_n - P\|$$

$$\Rightarrow \|x_{n+1} - P\| = \|x_n - P\| \Rightarrow \|x_n - p\|$$

is non-increasing, hence $\{x_n\}$ is bounded.

Since X is uniformly convex and $\{x_n\}$ is bounded by the Banach Alaoglu theorem $\{x_n\}$ has a weakly Convergent subsequence.

Let $x_{n_k} \rightarrow x^*$ (weak convergence) for some sequence$\{ x_{n_k} \}$. Now, we show that $x^* \in F$. i.e. x^* is a common Fixed point of T, S, R from the D.D. iteration.

$$x_{n+1} - x_n = \alpha_n\left(T_{(x_n)} - x_n\right) + \beta_n\left(S_{(x_n)} - x_n\right) + \gamma_n\left(R_{(x_n)} - x_n\right)$$

since $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$

we obtain.

$\|T_{(x_n)} - x_n\| \rightarrow 0, \ \ \|S_{(x_n)} - x_n\| \rightarrow 0, \ \ \|R_{(x_n)} - x_n\| \rightarrow 0$

So that

$$x_{n_k} \rightarrow x^* \ and \ \|T\left(x_{n_k}\right) - x_{n_k}\| \rightarrow 0$$

Since T is nonexpansive demiclosedness principle applies If $x_n \rightarrow x$ and $\|T_{(x_n)} - x_n\| \rightarrow 0$ then $x \in F(T)$ Thus, applying this principle to T, S, R we get

$$x^* \in F(T) \cap F(S) \cap F(R) = F$$

Suppose there is another weak subsequence limit $y^* \neq x^*$ with $x_{m_k} \rightarrow y^*$ and $y^* \in F$

Then by Opial's condition

$\lim_{n\to\infty} inf\|x_n - x^*\| \leq \lim_{n\to\infty} \inf \|x_n - y^*\|$ and

$\lim_{n\to\infty} \inf \|x_n - y^*\| < \lim_{n\to\infty} \inf \|x_n - x^*\|$

Which is a contradiction

Hence, the weak limit is unique and the full sequence $\{x_n\}$ converges weakly to this point. Therefore for the sequence $\{x_n\}$ generated by the D.D. iterative process converges weakly to a unique point $x^* \in F(T) \cap F(S) \cap F(R)$.

Theorem 3.2

Let X be a uniformly convex Banach space that satisfies Opial's Condition and let E ⊆ X be a nonempty, closed, convex and bounded subset. Suppose T, S and R : E → E are quasi non-expansive mapping with a common fixed point i.e. $F := F(T) \cap F(S) \cap F(R) \neq \phi$ where $F(T), F(S), F(R)$ denote the set of Fixed point of $T, S,$ and R respectively Let $\{x_n\}$ be a sequence in E generated by a D.D. Iteration process involving the mappings $T, S,$ and R. Then the sequence $\{x_n\}$ converges weakly to a point set F , the common fixed point set of T, S and R.

Proof.

Since E is bounded and the iteration is Quasi Non-expansive for each mapping.
i.e.
$\|T_x - P\| \leq \|x - P\|$ for any $P \in F$ similarly for S. R one shows directly that

$\|x_{n+1} - p\| \leq \|x_n - p\|$

Hence $\{\|x_n - p\|\}$ is monotonic decreasing, thus converges to some limit for each fixed $P \in F$. In particular $\{x_1\}$ is bounded and $\lim_{n\to\infty} \|x_n - p\|$ exists for every $P \in F$.

One show that

$\lim_{n\to\infty} \|T_{(x_n)} - x_n\| = 0, \|S_{(x_n)} - x_n\| \to 0,$

$\|R_{(x_n)} - x_n\| \to 0$

This follows from the structure of D.D process and Quasi-nonexpansiveness combined with the fact that the distances $\|x_n - p\|$ converges.

By boundness and reflexivity (uniform Convexity implies reflexivity) there exists a weakly convergent subsequence $x_{n_j} \to q$

for some $q \in E$.
Becouse

$\|T_{x_{n_j}} - x_{n_j}\| \to 0(\, and\ like\ wise\ for\ S, R)$

apply the demiclosedness principle (i.e. Mapping

$I - T, I - S, I - R$ are demiclosed at zero in a uniformly convex Banach space) to deduce.

$q = T_q, q = S_q, q = R_q$, $So\ q \in F$

Suppose that another sub-sequence $x_n \to V \in F$. Since limits lim $\|x_n - p\|$ exist for all $P \in F$, one has

$$\lim_{n\to\infty} inf \|x_n - q\| = \lim_{n\to\infty} \|x_n - q$$

By Opial's Condition, of $v \neq q$, the lim inf $\|x_n - q\| < lim$ inf $\|x_n - q\|$,

a contradiction to the equalities above. Hence $v = q$ thus all weak cluster points coincide of follows that $x_n \to q$ in full.

Therefore, the full sequence x_n converges weakly to a point $q \in F(T) \cap F(S) \cap F(R)$,

Conclusion

In this paper, we have proposed and analyzed a new class of iterative schemes for approximating fixed points of non-expansive and quasi-nonexpansive mappings in Banach spaces. By employing the D–D iterative process, we established convergence theorems under suitable geometric assumptions and control conditions. The results demonstrate that the constructed sequence converges strongly to the fixed point of the mapping, thereby extending and generalizing several existing fixed-point theorems in the literature. This study provides a useful framework for further applications of iterative methods in functional analysis and optimization problems within Banach spaces.

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